

HALL ALGEBRAS ASSOCIATED TO TRIANGULATED CATEGORIES, II: ALMOST ASSOCIATIVITY

FAN XU

Dedicated to Professor YingBo Zhang.

ABSTRACT. By using the approach in [8] to Hall algebras arising in homologically finite triangulated categories, we find an ‘almost’ associative multiplication structure for indecomposable objects in a 2-periodic triangulated category. As an application, we give a new proof of the theorem of Peng and Xiao in [5] which provides a way of realizing symmetrizable Kac-Moody algebras and elliptic Lie algebras via 2-periodic triangulated categories.

INTRODUCTION

Let \mathcal{U} be the universal enveloping algebra of a simple Lie algebra of type A, D or E over the field of rational numbers \mathbb{Q} . There are many interesting results involving the categorification and the geometrization of \mathcal{U} . The work of Gabriel [1] strongly suggested the possibility of the categorification. He showed that there exists a bijection between isomorphism classes of all indecomposable modules over a hereditary algebra of Dynkin type and the positive roots of the corresponding semisimple Lie algebra. In [6], Ringel explicitly realized the positive part of \mathcal{U} through the Hall algebra approach. A different but somewhat parallel realization was given by Lusztig. He showed that the negative part of \mathcal{U} can be geometrically realized by using constructible functions on affine spaces of representations of a preprojective algebra in [3]. One may naturally consider to recover the whole Lie algebras and the whole (quantized) enveloping algebras [6].

Nakajima [4] showed that an arbitrarily large finite-dimensional quotient of \mathcal{U} can be realized in terms of the homology of a triple variety. A different construction was given by Lusztig in [3] in terms of constructible functions on the triple variety. On the other hand, Peng and Xiao [5] defined a Lie bracket between two isomorphism classes of indecomposable objects in a k -additive triangulated category with the translation functor T satisfying $T^2 = 1$ for a finite field k with the cardinality q . It induces a Lie algebra over $\mathbb{Z}/(q-1)$, while it is still unknown which associative multiplication induces the Lie bracket over $\mathbb{Z}/(q-1)$. However, it seems to be hopeless to realize the whole enveloping algebra by the constructions of Nakajima [4] and Lusztig [3].

Recently, Toën gave a multiplication formula which defines an associative algebra (called the derived Hall algebra) corresponding to a dg category [7]. In [8],

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we extended to prove that Toën's formula can be applied to define an associative algebra for any triangulated category with some homological finiteness conditions. Unfortunately, a 2-periodic triangulated category does not satisfy these homological finiteness conditions in general. Hence, Toën's formula can not supply the realization of quantum groups. However, the approach in [8] strongly suggests the possibility to construct an associative multiplication over $\mathbb{Z}/(q-1)$.

Inspired by the method discussed in [8], in this paper, we prove that there exists an 'almost' associative multiplication over $\mathbb{Z}[\frac{1}{q}]/(q-1)$ for isomorphism classes of indecomposable objects in a 2-periodic triangulated category (Corollary 1.8). The associativity of the multiplication heavily depends on the choice of structure constants (Hall numbers) for defining the multiplication. The key techniques in this paper are to substitute derived Hall numbers in [7] or [8] for Hall numbers in [5] and introduce new variants associated to indecomposable objects. As a direct application, we obtain a new proof of the theorem of Peng and Xiao in [5], i.e., Theorem 2.4 in Section 3.

1. THE 'ALMOST' ASSOCIATIVITY

Given a finite field k with q elements, let \mathcal{C}_2 be a k -additive triangulated category with the translation functor $T = [1]$ satisfying (1) the homomorphism space $\text{Hom}_{\mathcal{C}_2}(X, Y)$ for any two objects X and Y in \mathcal{C} is a finite dimensional k -space, (2) the endomorphism ring $\text{End}X$ for any indecomposable object X is finite dimensional local k -algebra and (3) $T^2 \cong 1$. Then the category \mathcal{C}_2 is called a 2-periodic triangulated category. For any $M \in \mathcal{C}_2$, we set $\underline{\dim}M$ to be the canonical image of M in the Grothendieck group of \mathcal{C}_2 . Throughout this paper, we assume \mathcal{C}_2 is proper, i.e., for any nonzero indecomposable object X in \mathcal{C}_2 , $\underline{\dim}X \neq 0$. By $\text{ind}\mathcal{C}_2$ we denote the set of representatives of isomorphism classes of all indecomposable objects in \mathcal{C}_2 . For any indecomposable object $X \in \mathcal{C}_2$, we set $d(X) = \dim_k(\text{End}X/\text{radEnd}X)$. Recall that (see [5, Lemma 8.1])

$$|\text{Aut}X| = |\text{radEnd}X|(q^{d(X)} - 1).$$

For any X, Y and Z in \mathcal{C}_2 , we will use fg to denote the composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, and $|A|$ to denote the cardinality of a finite set A . Given $X, Y, L \in \mathcal{C}_2$, put

$$W(X, Y; L) = \{(f, g, h) \in \text{hom}(X, L) \times \text{hom}(L, Y) \times (Y, X[1]) \mid$$

$$X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1] \text{ is a triangle}\}.$$

There is a natural action of $\text{Aut}X \times \text{Aut}Y$ on $W(X, Y; L)$. The orbit space is denoted by $V(X, Y; L)$. The orbit of $(f, g, h) \in W(X, Y; L)$ is denoted by $(f, g, h)^\wedge$. Then

$$(f, g, h)^\wedge = \{(af, gc^{-1}, ch(a[1])^{-1}) \mid (a, c) \in \text{Aut}X \times \text{Aut}Y\}.$$

We also write $F_{XY}^L = |V(X, Y; L)|$. Throughout this section, we fix a triple pair (X, Y, Z) such that X, Y, Z are nonzero indecomposable objects in \mathcal{C}_2 and none of the following conditions holds

- (1) $X \cong Z \cong Y[1]$,
- (2) $X \cong Y \cong Z[1]$,
- (3) $Y \cong Z \cong X[1]$.

Lemma 1.1. *Let X, Y, Z and $M \neq 0$ be in \mathcal{C}_2 with X, Y, Z indecomposable and $M \oplus Z \not\cong X \oplus Y$. Then we have*

$$\frac{F_{XY}^{M \oplus Z}}{|\text{Aut} Z|} \in \mathbb{Z}[\frac{1}{q}].$$

where $\mathbb{Z}[\frac{1}{q}]$ is the polynomial ring for $\frac{1}{q}$ with coefficients in \mathbb{Z} .

Proof. We define the action of $\text{Aut} Z$ on $V(X, Y; M \oplus Z)$ as follows. For any $\alpha = ((f_1 \ f_2), \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, h)^\wedge \in V(X, Y; M \oplus Z)$ and $d \in \text{Aut} Z$, define

$$d.((f_1 \ f_2), \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, h)^\wedge = ((f_1 \ f_2 d^{-1}), \begin{pmatrix} g_1 \\ dg_2 \end{pmatrix}, h)^\wedge.$$

The orbit space is denoted by $\overline{V}(X, Y; M \oplus Z)$. Let

$$G_{\overline{\alpha}} = \{d \in \text{Aut} Z \mid ((f_1 \ f_2 d^{-1}), \begin{pmatrix} g_1 \\ dg_2 \end{pmatrix}, h)^\wedge = ((f_1 \ f_2), \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, h)^\wedge\}.$$

Then by definition, $G_{\overline{\alpha}}$ is equal to

$$\{d \in \text{Aut} Z \mid ((f_1 \ f_2 d^{-1}), \begin{pmatrix} g_1 \\ dg_2 \end{pmatrix}, h) = ((af_1 \ af_2), \begin{pmatrix} g_1 b^{-1} \\ g_2 b^{-1} \end{pmatrix}, bh(a[1])^{-1})$$

$$\text{for some } (a, b) \in \text{Aut} X \times \text{Aut} Y\},$$

or equivalently, equal to

$$\{d \in \text{Aut} Z \mid dg_2 = g_2 b, g_1 b = g_1, af_1 = f_1 \text{ and } f_2 d = af_2 \\ \text{for some } (a, b) \in \text{Aut} X \times \text{Aut} Y\}.$$

Given $(b, d) \in \text{Aut} Y \times \text{Aut} Z$ such that $dg_2 = g_2 b$ and $g_1 b = g_1$, we have the following diagram with the middle square being commutative

$$\begin{array}{ccccccc} X & \xrightarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} & M \oplus Z & \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} & Y & \xrightarrow{h} & X[1] \\ & & \uparrow \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} & & \uparrow b & & \\ X & \xrightarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} & M \oplus Z & \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} & Y & \xrightarrow{h} & X[1] \end{array}$$

By the axioms of triangulated categories, there exists $a \in \text{Aut} X$ such that $af_1 = f_1$ and $f_2 d = af_2$. Hence,

$$G_{\overline{\alpha}} = \{d \in \text{Aut} Z \mid dg_2 = g_2 b \text{ and } g_1 b = g_1 \text{ for some } b \in \text{Aut} Y\}.$$

The map g_1 naturally induces a triangle

$$M \xrightarrow{g_1} Y \xrightarrow{h_1} C(g_1) \longrightarrow M[1]$$

where $C(g_1)$ is the cone of the map g_1 . Let $b' = 1 - b$. Then we have

$$\{b' \mid b \in \text{Aut} Y \mid g_1 b = g_1\} = \{b' \in \text{End} Y \mid b' \in h_1 \text{Hom}(C(g_1), Y), 1 - b' \in \text{Aut} Y\}.$$

We claim that for any $t \in \text{Hom}(C(g_1), Y)$, both th_1 and h_1t are nilpotent. Assume that h_1t is not nilpotent, then it is invertible since Y is indecomposable. This implies $g_1 = 0$. Consider the following diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\quad} & M & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & M[1] \\ \vdots \downarrow u & & \downarrow v & & \downarrow & & \\ X & \xrightarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} & M \oplus Z & \xrightarrow{\begin{pmatrix} 0 \\ g_2 \end{pmatrix}} & Y & \xrightarrow{h} & X[1] \end{array}$$

where $v = \begin{pmatrix} 1 & 0 \end{pmatrix}$. The middle square is commutative, then there exists $u : M \rightarrow X$ such that the diagram is commutative. This implies $uf_1 = 1_M$. However, X is indecomposable and then u is an isomorphism. The morphism g_2 is also an isomorphism by the octahedral axiom as showed in the following diagram.

$$\begin{array}{ccccccc} & & Z & \xlongequal{\quad} & Z & & \\ & & \downarrow & & \downarrow g_2 & & \\ X & \xrightarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} & M \oplus Z & \xrightarrow{\begin{pmatrix} 0 \\ g_2 \end{pmatrix}} & Y & \xrightarrow{h} & X[1] \\ \parallel & & \downarrow & & \downarrow & & \\ X & \xrightarrow{f_1} & M & \xrightarrow{\quad} & C(g_2) & \xrightarrow{\quad} & X[1] \end{array}$$

where $C(g_2)$ is the cone of g_2 . It contradicts to the assumption $M \oplus Z \cong X \oplus Y$. Hence, $g_1 \neq 0$. In the same way, we have $g_2 \neq 0$. This implies that for any $t \in \text{Hom}(C(g_1), Y)$, both th_1 and h_1t are nilpotent. We have $1 - h_1t \in \text{Aut}Y$. Therefore, $G_{\bar{\alpha}}$ is isomorphic to

$$G'_{\bar{\alpha}} = \{d' \in \text{End}Z \mid 1 - d' \in \text{Aut}Z, d'g_2 = g_2b' \text{ for some } b' \in h_1\text{Hom}(C(g_1), Y)\}.$$

Let $d' \in \text{End}Z$ satisfy $d'g_2 = g_2b'$ for some $b' \in h_1\text{Hom}(C(g_1), Y)$. Since b' is nilpotent, we assume $(b')^k = 0$ for some $k \in \mathbb{N}$, then $(d')^k g_2 = 0$. However, Z is indecomposable and $g_2 \neq 0$. We deduce that d' is nilpotent and then $1 - d' \in \text{Aut}Z$. Hence, $G'_{\bar{\alpha}}$ is a vector space. Finally, we obtain

$$\frac{F_{XY}^{M \oplus Z}}{|\text{Aut}Z|} = \sum_{\bar{\alpha} \in \bar{V}(X, Y; M \oplus Z)} \frac{1}{|G_{\bar{\alpha}}|} \in \mathbb{Z}\left[\frac{1}{q}\right].$$

□

Note that the conclusion of the lemma may not hold if $M \oplus Z \cong X \oplus Y$ in Lemma 1.1, then . Indeed, if $X \not\cong Y$, $F_{XY}^{X \oplus Y} = |\text{Hom}(X, Y)| = q^n$ where $n = \dim_k \text{Hom}(X, Y)$. If $X \cong Y$, $F_{XY}^{X \oplus Y} = |\text{End}X| + |\text{radEnd}X|$.

Denote by $(X, Y)_Z$ the subset of $\text{Hom}_{\mathcal{C}_2}(X, Y)$ consisting of the morphisms whose cones are isomorphic to Z . The following proposition ([8, Proposition 2.5]) also holds for 2-periodic triangulated categories.

Proposition 1.2. For any $Z, L, M \in \mathcal{C}_2$, we have

(1) Any $\alpha = (l, m, n)^\wedge \in V(Z, L; M)$ has the representative of the form:

$$Z \xrightarrow{\begin{pmatrix} 0 \\ l_2 \end{pmatrix}} M \xrightarrow{\begin{pmatrix} 0 & m_2 \end{pmatrix}} L \xrightarrow{\begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}} Z[1]$$

where $Z = Z_1(\alpha) \oplus Z_2(\alpha)$, $L = L_1(\alpha) \oplus L_2(\alpha)$, n_{11} is an isomorphism between $L_1(\alpha)$ and $Z_1(\alpha)[1]$ and $n_{22} \in \text{radHom}(L_2(\alpha), Z_2(\alpha)[1])$.

$$\frac{|(M, L)_{Z[1]}|}{|\text{Aut} L|} = \sum_{\alpha \in V(Z, L; M)} \frac{|\text{End} L_1(\alpha)|}{|n(\alpha) \text{Hom}(Z[1], L)| |\text{Aut} L_1(\alpha)|}$$

and

$$\frac{|(Z, M)_L|}{|\text{Aut} Z|} = \sum_{\alpha \in V(Z, L; M)} \frac{|\text{End} Z_1(\alpha)|}{|\text{Hom}(Z[1], L) n(\alpha)| |\text{Aut} Z_1(\alpha)|}$$

$$\text{where } n(\alpha) = \begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}.$$

For $\alpha \in (l, m, n)^\wedge$, define

$$s(\alpha) = \dim_k n \text{Hom}(Z[1], L), \quad t(\alpha) = \dim_k \text{Hom}(Z[1], L) n.$$

For any objects U, V and W in \mathcal{C}_2 , define

- (1) $g_{UV}^W := \frac{|(U, W)_V|}{|\text{Aut} U|}$, if $U \not\cong W \oplus V[1]$,
- (2) $\bar{g}_{UV}^W := \frac{|(W, V)_{U[1]}|}{|\text{Aut} V|}$, if $V \not\cong W \oplus U[1]$,
- (3) $g_{W \oplus V[1], V}^W := |\text{Hom}(W, V[1])| \cdot \frac{|(W \oplus V[1], W)_V|}{|\text{Aut}(W \oplus V[1])|} = \frac{1}{|\text{Aut} V|}$,
- (4) $\bar{g}_{U, W \oplus U[1]}^W := |\text{Hom}(U[1], W)| \cdot \frac{|(W, W \oplus U[1])_{U[1]}|}{|\text{Aut}(W \oplus U[1])|} = \frac{1}{|\text{Aut} U|}$.

Different from [5], we will consider the image of numbers in $\mathbb{Z}[\frac{1}{q}]/(q-1)$ instead of $\mathbb{Z}/(q-1)$ where $\mathbb{Z}[\frac{1}{q}]$ is the polynomial ring for $\frac{1}{q}$ with coefficient in \mathbb{Z} .

We have the following corollary of Proposition 1.2.

Proposition 1.3. Let X, Y, Z, L, L' and M be in \mathcal{C}_2 with X, Y, Z and $M \neq 0$ being indecomposable. Then we have the following properties.

- (1) If $L \not\cong M \oplus Z[1]$ and $L \neq 0$, then the number $g_{XY}^L g_{ZL}^M$ belongs to $\mathbb{Z}[\frac{1}{q}]$.
- (2) If $L \cong M \oplus Z[1]$ and $L \not\cong X \oplus Y$, then the number $g_{XY}^L g_{ZL}^M$ belongs to $\mathbb{Z}[\frac{1}{q}]$.
- (3) If $L' \not\cong M \oplus Y[1]$ and $L' \neq 0$, then the number $g_{ZX}^{L'} g_{LY}^M$ belongs to $\mathbb{Z}[\frac{1}{q}]$.
- (4) If $L' \cong M \oplus Y[1]$ and $L' \not\cong X \oplus Z$, then $g_{ZX}^{L'} g_{LY}^M$ belongs to $\mathbb{Z}[\frac{1}{q}]$.
- (5) If $X \not\cong Y$ and $X \not\cong Y[1]$, then the numbers $g_{XY}^{X \oplus Y} g_{X[1], X \oplus Y}^Y - g_{X[1], X}^0 g_{0, Y}^Y$ and $g_{XY}^{X \oplus Y} g_{Y[1], X \oplus Y}^X - g_{Y[1], X}^{X \oplus Y[1]} g_{X \oplus Y[1], Y}^X$ belong to $\mathbb{Z}[\frac{1}{q}]$.
- (6) If $Z \not\cong X$ and $Z \not\cong X[1]$, then the number $g_{ZX}^{Z \oplus X} g_{Z \oplus X, X[1]}^Z - g_{X, X[1]}^0 g_{Z, 0}^Z$ and $g_{ZX}^{Z \oplus X} g_{Z \oplus X, Z[1]}^X - g_{X, Z[1]}^{X \oplus Z[1]} g_{Z, X \oplus Z[1]}^X$ belong to $\mathbb{Z}[\frac{1}{q}]$.
- (7) $g_{XY}^L g_{ZL}^M - g_{XY}^L \bar{g}_{ZL}^M \in \mathbb{Z}[\frac{1}{q}]$. In $\mathbb{Z}[\frac{1}{q}]/(q-1)$, $g_{XY}^L g_{ZL}^M - g_{XY}^L \bar{g}_{ZL}^M = 0$.
- (8) $g_{ZX}^{L'} g_{LY}^M - \bar{g}_{ZX}^{L'} g_{LY}^M \in \mathbb{Z}[\frac{1}{q}]$. In $\mathbb{Z}[\frac{1}{q}]/(q-1)$, $g_{ZX}^{L'} g_{LY}^M - \bar{g}_{ZX}^{L'} g_{LY}^M = 0$.

Proof. If $L \not\cong M \oplus Z[1]$, then by Proposition 1.2, we have $L_1(\alpha) = 0$ and

$$g_{ZL}^M = \sum_{\alpha \in V(Z, L; M)} \frac{1}{|n(\alpha) \text{Hom}(Z[1], L)|} \in \mathbb{Z}[\frac{1}{q}].$$

In the same way, $g_{XY}^L \in \mathbb{Z}[\frac{1}{q}]$. This proves (1). Next, we prove (2). In this case, $g_{ZL}^M = \frac{1}{|\text{Aut}Z|}$. As in the proof of Lemma 1.1, $g_{XY}^{M \oplus Z[1]}$ is equal to

$$\sum_{\alpha \in V(X,Y;M \oplus Z[1])} \frac{1}{|h(\alpha)\text{Hom}(X[1],Y)|} = \sum_{\bar{\alpha} \in \bar{V}(X,Y;M \oplus Z[1])} \frac{1}{|h(\alpha)\text{Hom}(X[1],Y)|} \frac{|\text{Aut}Z|}{|G_{\bar{\alpha}}|}.$$

This proves (2). The proofs of (3) and (4) are similar. As for (5), the number $g_{XY}^{X \oplus Y} g_{X[1],X \oplus Y}^Y - g_{X[1],X}^0 g_{0,Y}^Y$ is equal to

$$\frac{|\text{Hom}(X,Y)| - 1}{|\text{Aut}X|} = \frac{q^{\dim_k \text{Hom}(X,Y)} - 1}{|\text{radEnd}X|(q^{d(X)} - 1)}.$$

Since $\frac{\dim_k \text{Hom}(X,Y)}{d(X)} = l_{\text{End}X}(\text{Hom}(X,Y)) \in \mathbb{Z}$, the number belongs to $\mathbb{Z}[\frac{1}{q}]$. The proofs of the rest part of (5) and (6) are similar. We prove (7). If $L = 0$, then $g_{XY}^L g_{ZL}^M - g_{XY}^L \bar{g}_{ZL}^M = \frac{1}{|\text{Aut}X|} - \frac{1}{|\text{Aut}X|} = 0$. If $L \cong M \oplus Z[1]$, then $g_{ZL}^M = \bar{g}_{ZL}^M = \frac{1}{|\text{Aut}Z|}$. Hence, $g_{XY}^L g_{ZL}^M - g_{XY}^L \bar{g}_{ZL}^M = 0$. If $L \not\cong M \oplus Z[1]$ and $L \neq 0$, then by Proposition 1.2, we have $g_{XY}^L \in \mathbb{Z}[\frac{1}{q}]$ and $g_{ZL}^M - \bar{g}_{ZL}^M = \sum_{\alpha \in V(Z,L;M)} (\frac{1}{q^{s(\alpha)}} - \frac{1}{q^{t(\alpha)}}) \in \mathbb{Z}[\frac{1}{q}]$. This proves (7). The proof of (8) is similar. \square

Corollary 1.4. Let X, Y, Z and M be in \mathcal{C}_2 with X, Y, Z and $M \neq 0$ being indecomposable. Then we have

$$\sum_{[L], L \in \mathcal{C}_2} g_{XY}^L g_{ZL}^M - \sum_{[L'], L' \in \mathcal{C}_2} g_{ZX}^{L'} g_{L'Y}^M \in \mathbb{Z}[\frac{1}{q}]$$

and in $\mathbb{Z}[\frac{1}{q}]/(q-1)$,

$$\sum_{[L], L \in \mathcal{C}_2} g_{XY}^L g_{ZL}^M - \sum_{[L'], L' \in \mathcal{C}_2} g_{ZX}^{L'} g_{L'Y}^M = \sum_{[L], L \in \mathcal{C}_2} g_{XY}^L \bar{g}_{ZL}^M - \sum_{[L'], L' \in \mathcal{C}_2} \bar{g}_{ZX}^{L'} g_{L'Y}^M.$$

Proof. As for the first statement of the corollary, by Proposition 1.3 (1) and (2), it is sufficient to prove that the number

$$g_{XY}^{X \oplus Y} g_{Z, X \oplus Y}^M + \delta_{X,Y[1]} g_{XY}^0 g_{Z,0}^M - g_{ZX}^{Z \oplus X} g_{Z \oplus X, Y}^M - \delta_{X,Z[1]} g_{ZX}^0 g_{0,Y}^M \in \mathbb{Z}[\frac{1}{q}].$$

If $X \oplus Y \not\cong M \oplus Z[1]$ and $Z \oplus X \not\cong M \oplus Y[1]$, by Proposition 1.3 (1) and (3), this number belongs to $\mathbb{Z}[\frac{1}{q}]$. Hence, we only need to check the following cases:

- (1) $M \cong X, Y \cong Z[1], X \not\cong Y, X \not\cong Y[1]$,
- (2) $M \cong Y, X \cong Z[1], Y \not\cong X, Y \not\cong X[1]$,
- (3) $M \cong Z, X \cong Y[1], Z \not\cong X, Z \not\cong X[1]$.

We note that the cases (1),(2) and (3) are symmetric to each other. Proposition 1.3 (5) and (6) correspond to the cases (1),(2) and (3), respectively. This proves the first statement of the corollary. The second statement of the corollary can be deduced by the first statement and Proposition 1.3 (7) (8). \square

Here we recall some notations in [8]. Let X, Y, Z, L, L' and M be in \mathcal{C}_2 . Then define

$$\begin{aligned} \text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]} &:= \left\{ \begin{pmatrix} m \\ f \end{pmatrix} \in \text{Hom}(M \oplus X, L) \mid \right. \\ &\quad \left. \text{Cone}(f) \simeq Y, \text{Cone}(m) \simeq Z[1] \text{ and } \text{Cone} \begin{pmatrix} m \\ f \end{pmatrix} \simeq L'[1] \right\} \end{aligned}$$

and

$$\text{Hom}(L', M \oplus X)_L^{Y, Z[1]} := \{(f', -m') \in \text{Hom}(L', M \oplus X) \mid \text{Cone}(f') \simeq Y, \text{Cone}(m') \simeq Z[1] \text{ and } \text{Cone}(f', -m') \simeq L\}.$$

The orbit space of $\text{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}$ under the action of $\text{Aut} L$ is just the orbit space of $\text{Hom}(L', M \oplus X)_L^{Y, Z[1]}$ under the action of $\text{Aut} L'$ (see [8]). We denote by $V(L', L; M \oplus X)_{Y, Z[1]}$ the orbit space. The following diagram illustrates the relation among $V(X, Y; L), V(Z, L; M), V(Z, X; L'), V(L', Y; M)$ and $V(L', L; M \oplus X)_{Y, Z[1]}$.

$$(1.1) \quad \begin{array}{ccccccc} Z & \xlongequal{\quad} & Z & & & & \\ \downarrow l' & & \downarrow l & & & & \\ L' & \xrightarrow{\quad f' \quad} & M & \xrightarrow{\quad g' \quad} & Y & \xrightarrow{\quad h' \quad} & L'[1] \\ \downarrow m' & & \downarrow m & & \parallel & & \downarrow m'[1] \\ X & \xrightarrow{\quad f \quad} & L & \xrightarrow{\quad g \quad} & Y & \xrightarrow{\quad h \quad} & X[1] \\ \downarrow n' & & \downarrow n & & & & \\ Z[1] & \xlongequal{\quad} & Z[1] & & & & \end{array}$$

Lemma 1.5. *Let $\alpha \in V(L', L; M \oplus X)_{Y, Z[1]}$ has the representative of the form as follows:*

$$L'_1 \oplus L'_2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ f' & -m' \end{pmatrix}} M \oplus X \xrightarrow{\begin{pmatrix} 0 & m \\ 0 & f \end{pmatrix}} L_1 \oplus L_2 \xrightarrow{\begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix}} L'_1[1] \oplus L'_2[1]$$

such that θ_1 is a nonzero isomorphism where $L = L_1 \oplus L_2$ and $L' = L'_1 \oplus L'_2$. Then we have

$$M \cong X, \quad Z \cong Y[1], \quad L \cong X \oplus Y, \quad L' \cong X \oplus Y[1].$$

Proof. By definition, the triangle α induces the triangles

$$X \xrightarrow{\begin{pmatrix} 0 & f \end{pmatrix}} L_1 \oplus L_2 \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} Y \longrightarrow X[1]$$

and

$$L'_1 \oplus L'_2 \xrightarrow{\begin{pmatrix} 0 \\ f' \end{pmatrix}} M \longrightarrow Y \longrightarrow L'_1[1] \oplus L'_2[1].$$

Consider the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\begin{pmatrix} 0 & f \end{pmatrix}} & L_1 \oplus L_2 & \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} & Y & \longrightarrow & X[1] \\ \downarrow & & \downarrow & & \vdots v & & \\ 0 & \longrightarrow & L_1 & \xrightarrow{1} & L_1 & \longrightarrow & 0 \end{array}$$

The left square is commutative. Then there exists a map v such that $g_2 v = 1$. Since Y is indecomposable, we deduce $Y \cong L_1$ and $X \cong L_2$. In the same way,

we deduce $L'_1 \cong Y[1]$, $L'_2 \cong M$. If we consider the other two induced triangles, we obtain $L_1 \cong Z[1]$, $L_2 \cong M$ and $L'_1 \cong Z$, $L'_2 \cong X$. \square

We define the action of $\text{Aut}X$ on $V(L', L; M \oplus X)_{Y, Z[1]}$ by

$$a \cdot ((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta)^\wedge = ((f', -m'a^{-1}), \begin{pmatrix} m \\ af \end{pmatrix}, \theta)^\wedge$$

for any $a \in \text{Aut}X$ and $((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta)^\wedge \in V(L', L; M \oplus X)_{Y, Z[1]}$. The orbit space is denoted by $\bar{V}(L', L; M \oplus X)_{Y, Z[1]}$.

Lemma 1.6. *For any $\alpha \in V(L', L; M \oplus X)_{Y, Z[1]}$, the stable subgroup G_α of α under the action of $\text{Aut}X$ is isomorphic to a vector space if $L \neq 0$ and $L' \neq 0$.*

Proof. By definition,

$$G_\alpha = \{a \in \text{Aut}X \mid af = fb \text{ for some } b \in \text{Aut}L \text{ such that } mb = m\}.$$

- (1) If $f = 0$, then $Y \cong L \oplus X[1]$. However, Y is indecomposable, so $L = 0$. Hence, $Y \cong X[1]$ and $Z \cong M$. This imply $L' \cong Z \oplus X$. In the following, we assume $f \neq 0$.
(2) If $L \not\cong M \oplus Z[1]$, then $mb = b$ implies $b = 1 - b'$ with $b' \in n\text{hom}(Z[1], L)$ nilpotent. In this case, G_α is isomorphic to

$$G'_\alpha = \{a' \in \text{End}X \mid 1 - a' \in \text{Aut}X, a'f = fb' \text{ for some } b' \in n\text{hom}(Z[1], L)\}.$$

Since b' is nilpotent, a' is nilpotent. And since X is indecomposable, $1 - a' \in \text{Aut}X$. We claim that G'_α is a vector space. Indeed, $0 \in G'_\alpha$. For any $a'_1, a'_2 \in G'_\alpha$, we have $a'_i f = fb'_i$ for $i = 1, 2$ and some $b'_1, b'_2 \in n\text{hom}(Z[1], L)$. So $(a'_1 + a'_2)f = f(b'_1 + b'_2)$. It is clear that $b'_1 + b'_2 \in n\text{hom}(Z[1], L)$ and $1 - (a'_1 + a'_2) \in \text{Aut}X$ since $a'_1 + a'_2$ is nilpotent. This shows $a'_1 + a'_2 \in G'_\alpha$.

- (3) If $L \cong M \oplus Z[1]$, α has the representative of the following form:

$$L' \xrightarrow{\begin{pmatrix} f' & -m' \end{pmatrix}} M \oplus X \xrightarrow{\begin{pmatrix} 1 & 0 \\ f_1 & f_2 \end{pmatrix}} M \oplus Z[1] \xrightarrow{\theta} L'[1].$$

The stable subgroup is

$$G_\alpha = \{a \in \text{Aut}X \mid af = fb \text{ for some } b = \begin{pmatrix} 1 & 0 \\ b_{21} & b_{22} \end{pmatrix} \in \text{Aut}L\}$$

where $f = (f_1, f_2)$. Equivalently, we have

$$af_1 = f_1 + f_2 b_{21} \text{ and } af_2 = f_2 b_{22}.$$

Set $a' = a - 1$ and $b'_{22} = b_{22} - 1$. If $a' \in \text{Aut}X$, then

$$a' \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(b'_{22})^{-1}b_{21} & 1 \end{pmatrix} = \begin{pmatrix} 0 & a'f_2 \end{pmatrix}.$$

Since Y is indecomposable, this shows $X \cong Z[1]$ and $Y \cong M$. In this case, $L' = 0$. \square

For any $X \in \mathcal{C}_2$, we denote by u_X by the isomorphism class of X . Define the multiplication by

$$u_X u_Y := \sum_{[L]} g_{YX}^L u_L$$

and we set $u_X u_Y(L) = g_{XY}^L$.

Theorem 1.7. *Let X, Y, Z and $M \neq 0$ be indecomposable objects in \mathcal{C}_2 . Then in $\mathbb{Z}[\frac{1}{q}]/(q-1)$ we have*

(1) *If $X \cong Y[1], X \not\cong Z, X \not\cong Z[1]$ and $M \cong Z$, then*

$$[(u_Y u_X)u_Z - u_Y(u_X u_Z)](M) = -\frac{\dim_k \operatorname{Hom}(Z, X)}{d(X)}.$$

(2) *If $X \cong Z[1], X \not\cong Y, X \not\cong Y[1]$ and $M \cong Y$, then*

$$[(u_Y u_X)u_Z - u_Y(u_X u_Z)](M) = \frac{\dim_k \operatorname{Hom}(X, Y)}{d(X)}.$$

(3) *Otherwise,*

$$[(u_Y u_X)u_Z - u_Y(u_X u_Z)](M) = 0.$$

Proof. It is equivalent to compute the number $\sum_{[L], L \in \mathcal{C}_2} g_{XY}^L g_{ZL}^M - \sum_{[L'], L' \in \mathcal{C}_2} g_{ZX}^{L'} g_{L'Y}^M$. By Corollary 1.4, we have

$$\begin{aligned} & \sum_{[L], L \in \mathcal{C}_2} g_{XY}^L g_{ZL}^M - \sum_{[L'], L' \in \mathcal{C}_2} g_{ZX}^{L'} g_{L'Y}^M \\ = & \sum_{[L], L \in \mathcal{C}_2} g_{XY}^L \bar{g}_{ZL}^M - \sum_{[L'], L' \in \mathcal{C}_2} \bar{g}_{ZX}^{L'} g_{L'Y}^M \\ = & \sum_{[L], L \not\cong M \oplus Z[1] \in \mathcal{C}_2} \frac{|(X, L)_Y|}{|\operatorname{Aut} X|} \cdot \frac{|(M, L)_{Z[1]}|}{|\operatorname{Aut} L|} \\ & - \sum_{[L'], L' \not\cong M \oplus Y[1] \in \mathcal{C}_2} \frac{|(L', X)_{Z[1]}|}{|\operatorname{Aut} X|} \cdot \frac{|(L', M)_Y|}{|\operatorname{Aut} L'|} \\ & + |\operatorname{Hom}(Z[1], M)| \cdot \frac{|(X, M \oplus Z[1])_Y|}{|\operatorname{Aut} X|} \cdot \frac{|(M, M \oplus Z[1])_{Z[1]}|}{|\operatorname{Aut}(M \oplus Z[1])|} \\ & - |\operatorname{Hom}(M, Y[1])| \cdot \frac{|(M \oplus Y[1], X)_{Z[1]}|}{|\operatorname{Aut} X|} \cdot \frac{|(M \oplus Y[1], M)_Y|}{|\operatorname{Aut}(M \oplus Y[1])|} \\ = & \sum_{[L], [L']; L, L' \in \mathcal{C}_2} \frac{1}{|\operatorname{Aut} X|} \cdot \delta_{L, M \oplus Z[1]} \cdot \frac{|\operatorname{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}|}{|\operatorname{Aut} L|} \\ & - \sum_{[L], [L']; L, L' \in \mathcal{C}_2} \frac{1}{|\operatorname{Aut} X|} \cdot \delta'_{L', M \oplus Y[1]} \cdot \frac{|\operatorname{Hom}(L', M \oplus X)_L^{Y, Z[1]}|}{|\operatorname{Aut} L'|} \end{aligned}$$

where $\delta_{L, M \oplus Z[1]} = |\operatorname{Hom}(Z[1], M)|$ for $L \cong M \oplus Z[1]$ and 1 otherwise, $\delta_{L', M \oplus Y[1]} = |\operatorname{Hom}(M, Y[1])|$ for $L' \cong M \oplus Y[1]$ and 1 otherwise. By Proposition 1.2 and Lemma 1.5, unless $L \cong X \oplus Y$ and $L' \cong X \oplus Y[1]$, the above sum is equal to

$$\sum_{[L], [L']; L, L' \in \mathcal{C}_2} \left(\sum_{\alpha \in V(L', L; M \oplus X)_{Y, Z[1]}} \left(\frac{1}{q^{s(\alpha)}} - \frac{1}{q^{t(\alpha)}} \right) \right) \cdot \frac{1}{|\operatorname{Aut} X|}$$

where $s(\alpha) = \dim_k \theta \operatorname{Hom}(L'[1], L) - \delta_{L, M \oplus Z[1]} \dim_{\mathbb{C}} \operatorname{Hom}(Z[1], M)$ and

$$t(\alpha) = \dim_k \operatorname{Hom}(L'[1], L) \theta - \delta_{L', M \oplus Y[1]} \dim_{\mathbb{C}} \operatorname{Hom}(M, Y[1]).$$

Here, θ is a morphism from L to $L'[1]$ in a triangle belonging to α . Consider the action of $\operatorname{Aut} X$ on $V(L', L; M \oplus X)_{Y, Z[1]}$, we always have $s(\alpha) = s(a.\alpha)$ and

$t(\alpha) = t(a \cdot \alpha)$ for any $a \in \text{Aut} X$ and $\alpha \in V(L', L; M \oplus X)_{Y, Z[1]}$. Hence, the sum is again equal to

$$\begin{aligned} & \sum_{[L], [L']; L, L' \in \mathcal{C}_2} \left(\sum_{\bar{\alpha} \in \bar{V}(L', L; M \oplus X)_{Y, Z[1]}} \left(\frac{1}{q^{s(\alpha)}} - \frac{1}{q^{t(\alpha)}} \right) \cdot \frac{|\text{Aut} X|}{|G_\alpha|} \right) \cdot \frac{1}{|\text{Aut} X|} \\ &= \sum_{[L], [L']; L, L' \in \mathcal{C}_2} \left(\sum_{\bar{\alpha} \in \bar{V}(L', L; M \oplus X)_{Y, Z[1]}} \left(\frac{1}{q^{s(\alpha)}} - \frac{1}{q^{t(\alpha)}} \right) \cdot \frac{1}{|G_\alpha|} \right) \end{aligned}$$

where G_α is the stable subgroup for $\alpha \in V(L', L; M \oplus X)_{Y, Z[1]}$. If $L = 0, L' \cong X \oplus Z$ or $L \cong X \oplus Y, L' = 0$, then it is easy to know

$$\frac{|\text{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}|}{|\text{Aut} L|} = \frac{|\text{Hom}(L', M \oplus X)_L^{Y, Z[1]}|}{|\text{Aut} L'|} = 1.$$

Hence, if $X \cong Y[1], X \not\cong Z, X \not\cong Z[1]$ and $M \cong Z$ (i.e., $L = 0$ and $L' \cong X \oplus Z$), then

$$[(u_Y u_X) u_Z - u_Y (u_X u_Z)](M) = \frac{1}{|\text{Aut} X|} - \frac{1}{|\text{Aut} X|} \cdot |\text{Hom}(Z, X)|.$$

By [5, Lemma 8.1], we have

$$\frac{1}{|\text{Aut} X|} - \frac{1}{|\text{Aut} X|} \cdot |\text{Hom}(Z, X)| = -\frac{\dim_k \text{Hom}(Z, X)}{d(X)}.$$

If $X \cong Z[1], X \not\cong Y, X \not\cong Y[1]$ and $M \cong Y$ (i.e., $L \cong X \oplus Y$ and $L' = 0$), then

$$[(u_Y u_X) u_Z - u_Y (u_X u_Z)](M) = \frac{1}{|\text{Aut} X|} \cdot |\text{Hom}(X, Y)| - \frac{1}{|\text{Aut} X|}.$$

Otherwise, G_α is isomorphic to a vector space by Lemma 1.6. Therefore, the sum vanishes in $\mathbb{Z}[\frac{1}{q}]/(q-1)$. Now we assume that $L \cong X \oplus Y$ and $L' \cong X \oplus Y[1]$. We note that Lemma 1.5 shows that $M \cong X$ and $Z \cong Y[1]$ in this case. By Lemma 1.2, we have

$$\frac{|\text{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}|}{|\text{Aut} L|} = \frac{|V(L', L; M \oplus X)^{Y, Z[1]}|}{|\text{Aut} X| \cdot |\text{Hom}(Y, X)|}$$

and

$$\frac{|\text{Hom}(L', M \oplus X)_L^{Y, Z[1]}|}{|\text{Aut} L'|} = \frac{|V(L', L; M \oplus X)^{Y, Z[1]}|}{|\text{Aut} X| \cdot |\text{Hom}(X[1], Y)|}$$

for $L \cong X \oplus Y, L' \cong X \oplus Y[1], M \cong X$ and $Z \cong Y[1]$. In this case, we deduce that

$$|\text{Hom}(Z[1], M)| \cdot \frac{|V(L', L; M \oplus X)_{Y, Z[1]}|}{|\text{Aut} X| \cdot |\text{Hom}(Y, X)|} - |\text{Hom}(M, Y[1])| \cdot \frac{|V(L', L; M \oplus X)_{Y, Z[1]}|}{|\text{Aut} X| \cdot |\text{Hom}(X[1], Y)|}$$

vanishes. This completes the proof of the theorem. \square

We note that the theorem is a refinement of Proposition 7.5 in [5] which is a crucial point to prove the Jacobi identity in [5].

For any $X \in \text{ind} \mathcal{C}_2$, we introduce a new variant θ_X associated to X . Define a new multiplication between indecomposable objects by the following rules. For $X, Y \in \mathcal{C}_2$,

- (1) if $X \not\cong Y[1]$, $u_X \cdot u_Y := \sum_{[L]} g_{YX}^L u_L$,
- (2) $u_X \cdot u_{X[1]} := \sum_{[L]} g_{X[1]X}^L + \theta_{X[1]}$,

$$\begin{aligned} (3) \quad u_Y \cdot \theta_X &:= -\frac{\dim_k \operatorname{Hom}(X[1], Y)}{d(X)}, \\ (4) \quad \theta_X \cdot u_Y &:= -\frac{\dim_k \operatorname{Hom}(Y, X)}{d(X)}. \end{aligned}$$

Then we have the following direct corollary of Theorem 1.7.

Corollary 1.8. With the above notation, in $\mathbb{Z}[\frac{1}{q}]/(q-1)$, we have

$$[(u_Y \cdot u_X) \cdot u_Z - u_Y \cdot (u_X \cdot u_Z)](M) = 0.$$

for $M \neq 0$.

This shows that the new multiplication is ‘almost’ associative for indecomposable objects. Here, ‘almost’ means that $(u_Y \cdot u_X) \cdot u_Z$ or $u_Y \cdot (u_X \cdot u_Z)$ may be nonsense in $\mathbb{Z}[\frac{1}{q}]/(q-1)$, $(u_Y \cdot u_X) \cdot u_Z - u_Y \cdot (u_X \cdot u_Z)$ makes sense in $\mathbb{Z}[\frac{1}{q}]/(q-1)$.

2. THE THEOREM OF PENG AND XIAO

Let \mathcal{C}_2 be a 2-periodic triangulated category as in the last section. We denote by $G(\mathcal{C}_2)$ the Grothendieck group of \mathcal{C}_2 . For any $M \in \mathcal{C}_2$, we denote by h_M the canonical image of M in $G(\mathcal{C}_2)$, called the dimension vector of M . Denote by \mathbf{h} the subgroup of $\mathbb{Q} \otimes_{\mathbb{Z}} G(\mathcal{C}_2)$ generated by $\tilde{h}_M := \frac{h_M}{d(M)}$ for $M \in \operatorname{ind} \mathcal{C}_2$. Define a symmetric bilinear function $(- | -)$ on $\mathbf{h} \times \mathbf{h}$ as follows

$$(h_X | h_Y) = \dim_k \operatorname{Hom}(X, Y) - \dim_k \operatorname{Hom}(X, Y[1]) + \dim_k \operatorname{Hom}(Y, X) - \dim_k \operatorname{Hom}(Y, X[1])$$

for any $X, Y \in \mathcal{C}_2$. We note that $(\tilde{h}_X | h_Y) \in \mathbb{Z}[\frac{1}{q}]$ for indecomposable objects $X, Y \in \mathcal{C}_2$.

Let \mathbf{n} be a free abelian group with a basis $\{[X] \mid X \in \operatorname{ind} \mathcal{C}_2\}$. Let $\mathbf{g} = \mathbf{h} \oplus \mathbf{n}$ be a direct sum of \mathbb{Z} -modules. Consider the factor group $\mathbf{g}_{(q-1)} = \mathbf{g}/(q-1)\mathbf{g}$. We denote by u_M, h_M the corresponding residues. As we know, the associativity of the multiplication naturally deduces the Jacobi identity. Here, the situation is very similar. The ‘almost’ associativity of the multiplication defined by g_{XY}^L for indecomposable objects also deduces the Jacobi identity. For any indecomposable objects X, Y in \mathcal{C}_2 , we define the Lie bracket over $\mathbb{Z}[\frac{1}{q}]/(q-1)$ by

$$[u_X, u_Y] := [u_X, u_Y]_{\mathbf{n}} + \delta_{X, Y[1]} \tilde{h}_X.$$

where $[u_X, u_Y]_{\mathbf{n}} = \sum_{[L]} (g_{YX}^L - g_{XY}^L) u_L$. And,

$$[\tilde{h}_X, u_Y] = -[u_Y, \tilde{h}_X] = -(\tilde{h}_X | h_Y) u_Y \quad \text{and} \quad [\mathbf{h}, \mathbf{h}] = 0.$$

We introduce the notation

$$[u_X, u_Y]_{\mathbf{n}}(L) := g_{YX}^L - g_{XY}^L.$$

Lemma 2.1. [5, Lemma 7.4] *With the above notation, we have*

$$[u_X, u_Y]_{\mathbf{n}}(L) = 0$$

where L is a decomposable object in \mathcal{C}_2 .

By Lemma 2.1, the definition of Lie bracket is well defined.

Remark 2.2. (1) We refer to [2] to replace $(h_X | h_Y)$ in [5] by $(\tilde{h}_X | h_Y)$.

(2) In [5], the Lie bracket is defined by $[u_X, u_Y] := \sum_{[L] \in \operatorname{ind} \mathcal{C}_2} (F_{YX}^L - F_{XY}^L) u_L$.

However, in $\mathbb{Z}[\frac{1}{q}]/(q-1)$, $g_{YX}^L - g_{XY}^L = F_{YX}^L - F_{XY}^L$.

Proposition 2.3. Let X, Y, Z be indecomposable objects in \mathcal{C}_2 . Then in $\mathbb{Z}[\frac{1}{q}]/(q-1)$ we have

$$[[u_X, u_Y], u_Z] = [[u_X, u_Z], u_Y] + [u_X, [u_Y, u_Z]]$$

Proof. By definition, we know

$$[[u_X, u_Y], u_Z] = [[u_X, u_Y]_{\mathbf{n}}, u_Z]_{\mathbf{n}} - (g_{YX}^{Z[1]} - g_{XY}^{Z[1]})\tilde{h}_Z - \delta_{X,Y[1]}(h_X \mid h_Z)/d(X)u_Z.$$

By Lemma 2.1, we have

$$[u_X, u_Y]_{\mathbf{n}} = u_X u_Y - u_Y u_X$$

and

$$\begin{aligned} & [[u_X, u_Y]_{\mathbf{n}}, u_Z]_{\mathbf{n}}(M) \\ &= \sum_{L \in \text{ind}\mathcal{C}_2} (g_{YX}^L - g_{XY}^L)[u_L, u_Z](M) \\ &= \sum_{L \in \mathcal{C}_2} (g_{YX}^L - g_{XY}^L)[u_L, u_Z](M) \\ &= [(u_X u_Y)u_Z - (u_Y u_X)u_Z - u_Z(u_X u_Y) + u_Z(u_Y u_X)](M) \\ &\quad - g_{YX}^{M \oplus Z[1]}/|\text{Aut}Z| + g_{XY}^{M \oplus Z[1]}/|\text{Aut}Z| + g_{YX}^{M \oplus Z[1]}/|\text{Aut}Z| - g_{XY}^{M \oplus Z[1]}/|\text{Aut}Z| \\ &= [(u_X u_Y)u_Z - (u_Y u_X)u_Z - u_Z(u_X u_Y) + u_Z(u_Y u_X)](M). \end{aligned}$$

Let Δ be the set of the permutations of (X, Y, Z) . Then

$$\Delta = \{(X, Y, Z), (Y, X, Z), (Y, Z, X), (Z, Y, X), (X, Z, Y), (Z, X, Y)\}.$$

Define $\Delta^+ = \{(Y, X, Z), (Z, Y, X), (X, Z, Y)\}$ and $\Delta^- = \Delta \setminus \Delta^+$. If X, Y and Z satisfy one of the following properties

- (1) $X \cong Z \cong Y[1]$,
- (2) $X \cong Y \cong Z[1]$,
- (3) $Y \cong Z \cong X[1]$,

then it is clear that the identity in the lemma holds. If X, Y, Z and M satisfy the condition in Theorem 1.7 (3), then using Lemma 2.1 and Theorem 1.7, we have

$$\begin{aligned} & \{[[u_X, u_Z]_{\mathbf{n}}, u_Y]_{\mathbf{n}} + [u_X, [u_Y, u_Z]_{\mathbf{n}}]_{\mathbf{n}} - [[u_X, u_Y]_{\mathbf{n}}, u_Z]_{\mathbf{n}}\}(M) \\ &= \left\{ \sum_{(A,B,C) \in \Delta^+} ((u_A u_B)u_C - u_A(u_B u_C)) - \sum_{(A,B,C) \in \Delta^-} ((u_A u_B)u_C - u_A(u_B u_C)) \right\}(M) \\ &= \left\{ \sum_{(A,B,C) \in \Delta^+} [((u_A u_B)u_C - u_A(u_B u_C)) - ((u_C u_B)u_A - u_C(u_B u_A))] \right\}(M) \\ &= 0. \end{aligned}$$

It is enough to prove that

$$(2.1) \quad (g_{YX}^{Z[1]} - g_{XY}^{Z[1]})\tilde{h}_Z - (g_{ZX}^{Y[1]} - g_{XZ}^{Y[1]})\tilde{h}_Y - (g_{YZ}^{X[1]} - g_{ZY}^{X[1]})\tilde{h}_X = 0.$$

By the symmetry of the equation, we need to prove that

$$g_{YX}^{Z[1]}\tilde{h}_Z + g_{XZ}^{Y[1]}\tilde{h}_Y + g_{ZY}^{X[1]}\tilde{h}_X = 0.$$

By definition, we have, over $\mathbb{Z}[\frac{1}{q}]/(q-1)$,

$$\frac{g_{YX}^{Z[1]}}{d(Z)} = \frac{|(Y, Z[1])_X|}{|\text{Aut}Y| \cdot d(Z)} = \frac{|(Y[1], Z)_{X[1]}|}{|\text{Aut}Z| \cdot d(Y)} = \frac{g_{XZ}^{Y[1]}}{d(Y)}$$

and

$$\frac{g_{YX}^{Z[1]}}{d(Z)} = \frac{|(Z[1], X)_{Y[1]}|}{|\text{Aut } X| \cdot d(Z)} = \frac{|(Z, X)_Y|}{|\text{Aut } Z| \cdot d(X)} = \frac{g_{ZY}^{X[1]}}{d(X)}.$$

The equation (2.1) follows this and the property $h_Z + h_Y + h_X = 0$.

This shows that the Jacobi identity holds. Now we assume that the condition of Theorem 1.8 (1) holds, i.e., $X \cong Y[1]$ and $Z \cong M$ but $X \not\cong Z$ and $X \not\cong Z[1]$. We note that Lemma 2.1 and the properness condition imply $[u_X, u_Y]_n = 0$ if $X \cong Y[1]$. Following Theorem 1.7 (1), we have

$$[(u_X u_Y)u_Z - u_X(u_Y u_Z)](M) = -\frac{\dim_k \text{Hom}(Z, Y)}{d(X)}$$

and

$$[(u_Y u_X)u_Z - u_Y(u_X u_Z)](M) = -\frac{\dim_k \text{Hom}(Z, X)}{d(X)}.$$

By Lemma 2.1, this implies

$$[u_Y(u_X u_Z) - u_X(u_Y u_Z)](M) = \frac{\dim_k \text{Hom}(Z, X) - \dim_k \text{Hom}(Z, Y)}{d(X)}.$$

Similarly, using Theorem 1.7 (2), we have

$$[(u_Z u_Y)u_X - (u_Z u_X)u_Y](M) = -\frac{\dim_k \text{Hom}(X, Z) - \dim_k \text{Hom}(X, Z[1])}{d(X)}.$$

Therefore, we obtain

$$\{[[u_X, u_Z]_n, u_Y]_n + [u_X, [u_Y, u_Z]_n]_n\}(Z) = -(h_X \mid h_Z)/d(X)[[u_X, u_Y], u_Z](Z).$$

In the same way, the Jacobi identity holds for the case that $X \cong Z[1]$ and $Y \cong M$ but $X \not\cong Y$ and $X \not\cong Y[1]$. This completes the proof of the proposition. \square

The following is the main result in [5].

Theorem 2.4. *Endowed with the above Lie bracket, $\mathfrak{g}_{(q-1)}$ is a Lie algebra over $\mathbb{Z}/(q-1)$.*

Proof. Let X, Y and $Z \in \text{ind } \mathcal{C}_2$. By Remark 2.1, we know that in $\mathbb{Z}[\frac{1}{q}]/(q-1)$, $g_{YX}^L - g_{XY}^L = F_{YX}^L - F_{XY}^L$ and $F_{YX}^L - F_{XY}^L \in \mathbb{Z}$. By Proposition 2.3, we only need to verify the Jacobi identities

$$[[\tilde{h}_X, u_Y], u_Z] = [[\tilde{h}_X, u_Z], u_Y] + [\tilde{h}_X, [u_Y, u_Z]]$$

and

$$[[\tilde{h}_X, \tilde{h}_Y], u_Z] = [[\tilde{h}_X, u_Z], \tilde{h}_Y] + [\tilde{h}_X, [\tilde{h}_Y, u_Z]]$$

hold. They can be easily verified by the definition of Lie bracket. We refer to [5] or [2] for details. \square

By Corollary 1.8, we can give a new version of the above theorem. For any $M \in \mathcal{C}_2$, we set

$$\tilde{h}_M^* = \theta_M - \theta_{M[1]} \text{ and } h_M^* = d_M \tilde{h}_M^*.$$

It is clear that $\tilde{h}_M^* = -\tilde{h}_{M[1]}^*$. Let \mathbf{h}^* be the abelian group generated by \tilde{h}_M^* for $M \in \text{ind } \mathcal{C}_2$. The symmetric bilinear form for \mathbf{h} naturally induces the symmetric bilinear form for \mathbf{h}^* , denoted by $(- \mid -)^*$. Moreover, we have the following result analog to the additivity of dimension vectors in $G(\mathcal{C}_2)$.

Lemma 2.5. *Let $M \xrightarrow{f} L \xrightarrow{g} N \xrightarrow{h} M[1]$ be a triangle for M, N and L in \mathcal{C}_2 . Then for any $Z \in \mathcal{C}_2$, we have*

$$(h_M^* + h_N^* | h_Z^*)^* = (h_L^* | h_Z^*)^*.$$

Proof. Applying the functor $\text{Hom}(Z, -)$ on the triangle, we obtain a long exact sequence

$$\cdots \longrightarrow \text{Hom}(Z, N[-1]) \xrightarrow{h^*} \text{Hom}(Z, M) \longrightarrow \cdots$$

Then we have

$$\begin{aligned} \dim_k \text{Hom}(Z, M) - \dim_k \text{Hom}(Z, M[1]) + \dim_k \text{Hom}(Z, N) - \dim_k \text{Hom}(Z, N[1]) \\ = \dim_k \text{Hom}(Z, L) - \dim_k \text{Hom}(Z, L[1]). \end{aligned}$$

Applying the functor $\text{Hom}(-, Z)$ on the triangle, we obtain

$$\begin{aligned} \dim_k \text{Hom}(M, Z) - \dim_k \text{Hom}(M[1], Z) + \dim_k \text{Hom}(N, Z) - \dim_k \text{Hom}(N[1], Z) \\ = \dim_k \text{Hom}(L, Z) - \dim_k \text{Hom}(L[1], Z). \end{aligned}$$

This completes the proof of the proposition. \square

Lemma 2.6. *For any X, Y and $Z \in \text{ind}\mathcal{C}_2$, we have*

$$(\theta_X \cdot \theta_Y) \cdot u_Z = (\theta_Y \cdot \theta_X) \cdot u_Z \text{ and } u_Z \cdot (\theta_X \cdot \theta_Y) = u_Z \cdot (\theta_Y \cdot \theta_X).$$

Lemma 2.5 and 2.6 strongly suggest us to make the following assumptions for \mathbf{h}^* . For any X, Y and L in \mathcal{C}_2 with the triangle $X \rightarrow L \rightarrow Y \rightarrow X[1]$, we have $h_X^* + h_Y^* = h_L^*$ and $h_X^* h_Y^* = h_Y^* h_X^*$. Let $\mathbf{g}^* = \mathbf{h}^* \oplus \mathbf{n}$ and $\mathbf{g}_{(q-1)}^* = \mathbf{g}^*/(q-1)\mathbf{g}^*$. Then the new multiplication in the last section naturally induces a Lie bracket over $\mathbf{g}_{(q-1)}^*$, i.e., for $X, Y \in \text{ind}\mathcal{C}_2$,

$$[u_X, u_Y]^* := u_X \cdot u_Y - u_Y \cdot u_X, \quad [\tilde{h}_X^*, u_Y]^* := \tilde{h}_X^* \cdot u_Y - u_Y \cdot \tilde{h}_X^*$$

and

$$[\tilde{h}_X^*, \tilde{h}_Y^*]^* := 0.$$

Then we have an analogue of Theorem 2.4 which is a direct consequence of Corollary 1.8 and the fact that the ‘almost’ associativity implies the Jacobi identity.

Theorem 2.7. *With the above notation and assumptions, $\mathbf{g}_{(q-1)}^*$ is a Lie algebra over $\mathbb{Z}/(q-1)$.*

Proof. Given indecomposable objects X, Y and Z in \mathcal{C}_2 , we need to prove the Jacobi identities

$$\begin{aligned} [[u_X, u_Y]^*, u_Z]^* &= [[u_X, u_Z]^*, u_Y]^* + [u_X, [u_Y, u_Z]^*]^*, \\ [[\tilde{h}_X^*, u_Y]^*, u_Z]^* &= [[\tilde{h}_X^*, u_Z]^*, u_Y]^* + [\tilde{h}_X^*, [u_Y, u_Z]^*]^* \end{aligned}$$

and

$$[[\tilde{h}_X^*, \tilde{h}_Y^*]^*, u_Z]^* = [[\tilde{h}_X^*, u_Z]^*, \tilde{h}_Y^*]^* + [\tilde{h}_X^*, [\tilde{h}_Y^*, u_Z]^*]^*$$

hold. The second and third identities follows Proposition 2.5. We prove the first identity. By Corollary 1.8, it is enough to prove

$$[[u_X, u_Y]^*, u_Z]^*(0) = [[u_X, u_Z]^*, u_Y]^*(0) + [u_X, [u_Y, u_Z]^*]^*(0).$$

The proof is similar to the proof of equation (2.1). \square

However, it is not clear whether there exists some explicit relation between $\mathfrak{g}_{(q-1)}^*$ and $\mathfrak{g}_{(q-1)}$.

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DEPARTMENT OF MATHEMATICS, TSINGHUA UNIVERSITY, BEIJING 100084, P.R.CHINA
E-mail address: fanxu@mail.tsinghua.edu.cn